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OPTIMAL CONTROL OF PROCESSES DESCRIBED BY EQUATIONS OF HYPERBOLIC TYPE

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Control problems are analyzed for systems described by second-order differential equations of hyperbolic type, set in the form of a connected multidimensional Bolza problem of the calculus of variations. Necessary stationarity conditions have been obtained. It is shown that to the optimal solutions there can correspond Lagrange multipliers which can have discontinuities on the characteristic hypersurfaces inside the domain.

Optimization problems for two-dimensional hyperbolic equations were examined earlier in [1—4]. Discontinuities in the Lagrange multipliers were first obtained when solving variational problems of gas dynamics [5].

1. Statement of the problem. Let us consider the partial differential equation and the relations

$$L(z) = \frac{\partial^2 z}{\partial x_0^2} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{i=0}^{n} a_i \frac{\partial z}{\partial x_i} = f(x_0, x, u, z) \tag{1.1}$$

$$\Psi_i(x_0, x, u) = 0, \quad i = 1, \ldots, m < p \tag{1.2}$$

$$u = (u_1(x_0, x), \ldots, u_p(x_0, x)), \quad a_{ij} = a_{ij}(x), \quad a_i = a_i(x)$$

given in an $(n+1)$-dimensional domain $W = \Omega \times [t_0, T]$. Here Ω is a finite connected domain of variation of the variables $x = (x_1, \ldots, x_n)$, $z(x_0, x)$ is a piecewise-smooth function of the variables x_0 and x, subject to definition, u is the p-dimensional vector of piecewise-continuous controls. The matrix composed from the coefficients a_{ij} is positive definite and symmetric at each point $x \in \Omega$. From these statements it follows that Eq. (1.1) is of hyperbolic type in W. The initial and boundary conditions

$$z\Big|_{x_0=t_0} = \varphi_1(x), \quad \frac{\partial z}{\partial x_0}\Big|_{x_0=t_0} = \varphi_2(x) \tag{1.3}$$

$$\frac{\partial z}{\partial N}\Big|_S = g(S, v, z|_S) \tag{1.4}$$

$$\psi_i(S, v) = 0, \quad i = 1, \ldots, m_1 < q; \quad v = (v_1(S), \ldots, v_q(S)) \tag{1.5}$$

are taken as specified. Here S is a cylindrical hypersurface formed by an $(n-1)$-dimensional surface Γ_0 moving along the coordinate x_0, $\partial z / \partial N$ is the derivative with respect to the normal to S, v is a q-dimensional vector of piecewise-continuous controls specified on S. The conditions

$$\chi_i\,(\Gamma, z|_{\Gamma_0}, z|_{\Gamma_1}, \ldots, z|_{\Gamma_r}) = 0, \quad i = 1, \ldots, m_2 \leqslant r+1 \tag{1.6}$$

can be specified at the final instant $x_0 = T$. Here Γ_i are ($n-1$)-dimensional surfaces lying in the hypersurface Ω, Γ_0 is the surface bounding Ω, Γ are generalized coordinates yielding all $\Gamma_i\,(i = 0, 1, \ldots, r)$. As an example we can cite the two circumferences $x_1 = r_1 \cos\varphi, x_2 = r_1 \sin\varphi$ and $x_1 = r_2 \cos\varphi, x_2 = r_2 \sin\varphi$, lying in the ($x_1, x_2$)-plane. Equalities (1.6) are to be understood as the connections between the values $z|_{\Gamma_i}$ for one and the same values of coordinates Γ.

We pose the following optimization problem: among the piecewise-continuous controls u and v satisfying relations (1.2) and (1.5) and the piecewise-smooth functions $z\,(x_0, x)$ satisfying Eq. (1.1), initial conditions (1.3), boundary conditions (1.4), and also the end conditions (1.6), find those which minimize the functional

$$I = \int_W f_0(x_0, x, u, z)\,dx_0\,dx + \int_S g_0(S, v, z|_S)\,dS + \tag{1.7}$$
$$\int_\Omega \varphi_0\left[x, z(T, x), \frac{\partial z}{\partial x_0}(T, x)\right] dx + \int_\Gamma \chi_0(\Gamma, z|_{\Gamma_0}, z|_{\Gamma_1}, \ldots, z|_{\Gamma_r})\,d\Gamma$$

The coefficients a_{ij} and a_i and the functions $f, f_0, \Psi_i, \varphi_1, \varphi_2, \psi_i, g, g_0, \chi_i$ and χ_0 are assumed to be continuous and to have continuous partial derivatives in all their arguments up to third order, inclusively, in the domain being examined. The function φ_0 is assumed continuous together with all its third partial derivatives in the subdomains of Ω delineated by the surfaces $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$.

2. Necessary stationary conditions. The problem formulated is an $(n+1)$-dimensional Bolza problem of the calculus of variations. For it we can prove a necessary condition for the stationarity of the functional

$$\delta I = 0 \tag{2.1}$$

in which δI is the first variation of the functional

$$I = \int_W [\lambda L(z) + H]\,dx_0\,dx + \int_S (\nu z_N|_S + h)\,dS + \tag{2.2}$$
$$\int_\Omega \{\nu_1[z(t_0, x) - \varphi_1(x)] + \nu_2[z_{x_0}(t_0, x) - \varphi_2(x)] +$$
$$\varphi_0\}\,dx + \int_\Gamma \chi\,d\Gamma, \quad H = f_0 - \lambda f + \sum_{i=1}^{m} \mu_i \Psi_i$$
$$h = g_0 - \nu g + \sum_{i=1}^{m_1} \varkappa_i \psi_i, \quad \chi = \chi_0 + \sum_{i=1}^{m_2} \rho_i \chi_i$$

where $\lambda\,(x_0, x)$, $\mu_i\,(x_0, x)$, $\varkappa_i\,(S)$, $\nu\,(S)$, $\nu_i\,(x)$ and $\rho_i\,(\Gamma)$ are undetermined Lagrange multipliers. When computing the first variation δI we assume that the whole domain W consists of a finite number of elementary domains W_i bounded by a piecewise-smooth hypersurface S_i whose smooth parts S_{ij} are separated by ($n-1$)-dimensional surfaces C_{ij}. Some parts of the hypersurfaces S_i can coincide with the boundaries of domain W: S and Ω (when $x_0 = t_0$ or $x_0 = T$), while some C_{ij} will coincide with surfaces Γ_i. In order not to obscure the calculations with indices showing to which specific elementary domains the quantities belong, we compute the first variation δI in some elementary domain w which has a boundary s and $(n-1)$-dimensional surfaces C. We assume that the elementary domain w is bounded by the hypersurfaces

$$y_0 = y_0(x_0, x) = D \tag{2.3}$$

where D is some arbitrary constant. Naturally, in the general case the form of (2.3) is different from the smooth segments of boundary s.

The first variation of functional I in domain w is

$$\delta I_w = \int_w \Big[\lambda L(\delta z) + \frac{\partial H}{\partial z}\delta z + \sum_{i=1}^{p}\frac{\partial H}{\partial u_i}\delta u_i\Big] dx_0\, dx + \int_s [\lambda L(z) + H]\,\delta y_0 ds \tag{2.4}$$

The first variations of the last three terms in (2.2), given on the boundaries of W, should be joined on the expression (2.4) (see below). We transform (2.4) by applying the following transformation to the terms $\lambda L(\delta z)$:

$$\lambda a_{ij}\frac{\partial^2 \delta z}{\partial x_i\, \partial x_j} = \frac{\partial}{\partial x_i}\Big(\lambda a_{ij}\frac{\partial \delta z}{\partial x_j}\Big) - \frac{\partial}{\partial x_j}\Big[\frac{\partial}{\partial x_i}(a_{ij}\lambda)\,\delta z\Big] + \frac{\partial^2 (a_{ij}\lambda)}{\partial x_i \partial x_j}\delta z, \quad \lambda a_i\frac{\partial \delta z}{\partial x_i} = \frac{\partial}{\partial x_i}(\lambda a_i \delta z) - \frac{\partial}{\partial x_i}(\lambda a_i)\,\delta z$$

Then we obtain

$$\delta I_w = \int_w \Big\{\Big[M(\lambda) + \frac{\partial H}{\partial z}\Big]\delta z + \sum_{i=1}^{p}\frac{\partial H}{\partial u_i}\delta u_i\Big\} dx_0\, dx + \tag{2.5}$$

$$\int_s \Big\{\lambda \alpha_{00}\frac{\partial \delta z}{\partial x_0} - \alpha_{00}\frac{\partial \lambda}{\partial x_0}\delta z - \sum_{i,j=1}^{n}\Big[\lambda a_{ij}\alpha_{0i}\frac{\partial \delta z}{\partial x_j} - \alpha_{0j}\frac{\partial (a_{ij}\lambda)}{\partial x_i}\delta z\Big] + \sum_{i=0}^{n}\lambda a_i \alpha_{0i}\delta z + f_0 \delta y_0\Big\} ds$$

$$M(\lambda) = \frac{\partial^2 \lambda}{\partial x_0^2} - \sum_{i,j=1}^{n}\frac{\partial^2 (a_{ij}\lambda)}{\partial x_i\, \partial x_j}$$

Here α_{0i} are the direction cosines of the normal to hypersurface (2.3), determined in the Appendix. Besides those written out in (2.5) there are no terms in the expression for δI, depending on the interior points of the elementary domain w. Therefore, following the formalism of the calculus of variations, it is necessary to set the conditions

$$M(\lambda) + \partial H / \partial z = 0 \tag{2.6}$$

$$\partial H / \partial u_i = 0, \quad i = 1, \ldots, p \tag{2.7}$$

which must be satisfied at each interior point of each elementary domain.

2. The Weierstrass — Erdmann conditions. To obtain the Weierstrass—Erdmann conditions on the boundary hypersurfaces s of the elementary domains we analyze the remainder terms in the first variation δI. To do this we pass in δI to the new curvilinear orthogonal coordinates

$$y_0 = y_0(x_0, x), \quad y_1 = y_1(x_0, x), \quad \ldots, \quad y_n = y_n(x_0, x) \tag{3.1}$$

$$\frac{\partial}{\partial x_i} = \sum_{j=0}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}, \qquad i = 0, 1, \ldots, n$$

Using (A.4), (A.5), (A.7) and (A.10) from the Appendix, we obtain

$$\delta I_w = \int_s \Big(b_0 \lambda \frac{\partial \delta z}{\partial y_0} + \lambda \sum_{k=1}^{n} b_k \frac{\partial \delta z}{\partial y_k} - b_0 \frac{\partial \lambda}{\partial y_0} \delta z - \tag{3.2}$$

$$\sum_{k=1}^{n} b_k \frac{\partial \lambda}{\partial y_k} \delta z + c_0' \lambda \delta z + d_0 \lambda \delta z + f_0 \delta y_0 \Big) \frac{dy}{R_0 R_1 \ldots R_n}$$

$$\Big(dy = dy_1 \ldots dy_n, \; c_0' = \sum_{i,j=1}^{n} \frac{\partial a_{ij}}{\partial x_i} \frac{\partial y_0}{\partial x_j} \Big)$$

We transform the second term in the integrand in (3.2) by using the formula

$$\frac{\partial}{\partial y_k} \Big(\frac{b_k \lambda \delta z}{R_0 R_1 \ldots R_n} \Big) = \frac{\partial}{\partial y_k} \Big(\frac{b_k \lambda}{R_0 R_1 \ldots R_n} \Big) \delta z + \frac{b_k \lambda}{R_0 R_1 \ldots R_n} \frac{\partial \delta z}{\partial y_k}$$

Then the first variation (3.2) becomes

$$\delta I_w = \int_s \Big[b_0 \lambda \frac{\partial \delta z}{\partial y_0} + F(\lambda) \delta z + f_0 \delta y_0 \Big] \frac{dy}{R_0 R_1 \ldots R_n} + \tag{3.3}$$

$$\int_C \sum_{k=1}^{n} b_k \cos(N, y_k) \frac{\lambda \delta z}{R_0 R_1 \ldots R_n} dC$$

$$F(\lambda) = - b_0 \frac{\partial \lambda}{\partial y_0} - 2 \sum_{k=1}^{n} b_k \frac{\partial \lambda}{\partial y_k} - \frac{\partial^2 y_0}{\partial x_0^2} \lambda + \frac{\partial b_0}{\partial y_0} \lambda +$$

$$2 c_0' \lambda + d_0 \lambda + \frac{b_0}{R_0} \sum_{k=0}^{n} \frac{1}{\rho_{0k}} \lambda$$

where $1 / \rho_{0k}$ are the quantities defined by formulas (A.9) in the Appendix, C is the ($n - 1$)-dimensional surface on which the smoothness of the hypersurface s is disrupted and N is the normal to surface C, lying in the tangent hyperplane to s.

We transform the variations $\delta \partial z / \partial y_0, \delta z$ and $\delta z |_C$ occurring in relation (3.3) by the formulas

$$\delta \frac{\partial z}{\partial y_0} = \Delta \frac{\partial z}{\partial y_0} - \frac{\partial^2 z}{\partial y_0^2} \delta y_0, \quad \delta z = \Delta z - \frac{\partial z}{\partial y_0} \delta y_0 \tag{3.4}$$

$$\delta z |_C = \Delta z |_C - \frac{\partial z}{\partial y_0} \delta y_0 - \frac{\partial z}{\partial N} \delta N$$

where Δz. and $\Delta \partial z / \partial y_0$ are variations on surface s with due regard to its mobility away from C and $\Delta z |_C$ is the variation with due regard to the mobility on surface C. Making use of equalities (3.4), we transform δI_w to the form

$$\delta I_w = \int_s \Big\{ b_0 \lambda \Delta \frac{\partial z}{\partial y_0} + F(\lambda) \Delta z + \Big[f_0 - \lambda f + 2\lambda \sum_{k=1}^{n} b_k \times \tag{3.5}$$

$$\frac{\partial^2 z}{\partial y_0 \partial y_k} + \lambda \frac{\partial^2 y_0}{\partial x_0^2} \frac{\partial z}{\partial y_0} + \lambda \sum_{k=1}^{n} \frac{\partial^2 y_k}{\partial x_0^2} \frac{\partial z}{\partial y_k} - \lambda \sum_{k,l=1}^{n} b_{kl} \frac{\partial^2 z}{\partial y_k \partial y_l} -$$

$$\lambda \sum_{k=0}^{n} c_k \frac{\partial z}{\partial y_k} + \lambda \sum_{k=1}^{n} d_k \frac{\partial z}{\partial y_k} - F(\lambda) \frac{\partial z}{\partial y_0} \Big] \delta y_0 \Big\} \frac{dy}{R_0 R_1 \ldots R_n} +$$

$$\int_C \sum_{k=1}^{n} b_k \cos(N, y_k)\, \lambda \left(\Delta z |_C - \frac{\partial z}{\partial y_0} \delta y_0 - \frac{\partial z}{\partial N} \delta N \right) \frac{dy}{R_0 R_1 \ldots R_n}$$

in which $\partial^2 z / \partial y_0^2$ is substituted from formula (A. 10) of the Appendix.

We go on to establish the Weierstrass — Erdmann conditions on the hypersurfaces s and on the ($n-1$)-dimensional surfaces C. We shall examine s and C lying inside W; therefore, s serves as the boundary of two elementary domains. We denote the quantities in the elementary domain being examined by the index plus and in a boundary elementary domain by the index minus. We note as well that $b_0 = 0$ is the differential equation determining the characteristic surfaces of Eqs. (1. 1) and (2. 6).

Let us obtain first of all the Weierstrass — Erdmann conditions on hypersurfaces s which are not characteristic ones. When passing through these hypersurfaces the quantities z and $\partial z / \partial y_0$ remain continuous, therefore

$$\Delta \frac{\partial z^+}{\partial y_0} = \Delta \frac{\partial z^-}{\partial y_0} = \Delta \frac{\partial z}{\partial y}, \quad \Delta z^+ = \Delta z^- = \Delta z, \quad \delta y_0^+ = \delta y_0^- = \delta y_0 \qquad (3.6)$$

Singling out in (3. 5) the terms containing $\Delta \partial z / \partial y_0$ and equating them to zero, we obtain $b_0 \lambda^+ - b_0 \lambda^- = 0$, $b_0 \neq 0$, therefore

$$\lambda^+ = \lambda^- \text{ on } s \qquad (3.7)$$

Singling out in (3. 5) the terms depending on Δz, we find

$$F(\lambda^+) = F(\lambda^-) \qquad (3.8)$$

The multiplier λ is continuous on hypersurfaces s, therefore, the derivatives $\partial \lambda / \partial y_k$ are continuous and from equality (3. 8) follows

$$\partial \lambda^+ / \partial y_0 = \partial \lambda^- / \partial y_0 \text{ on } s \qquad (3.9)$$

Equating the coefficients of variation δy_0 to zero and allowing for (3. 7) and (3. 9), we obtain

$$H^+ = H^- \text{ on } s \qquad (3.10)$$

Let us now obtain the Weierstrass — Erdmann conditions on hypersurfaces s which are characteristic. On characteristic hypersurfaces the derivative $\partial z / \partial y_0$ can have discontinuities as a consequence of discontinuities of the control functions $u\ (x_0, x)$ on them (see Eq. (A. 14)) or at the expense of discontinuities of the boundary controls v. We assume that the characteristics remain fixed in the problem at hand. Then $\delta y_0 = 0$ and $\delta N = 0$ and only the second relations in (3. 6) are valid. Since $b_0 = 0$ on the characteristics, the coefficients of the variations $\Delta \partial z / \partial y_0$ in (3. 5) equal zero and, therefore, we can have

$$\lambda^+ \neq \lambda^- \text{ on } s \qquad (3.11)$$

Equating the coefficient of variation Δz to zero, we obtain the condition

$$-2 \sum_{k=1}^{n} b_k \frac{\partial [\lambda]}{\partial y_k} + \left(c_0 + 2c_0' + d - \frac{\partial^2 y_0}{\partial x_0^2} + \frac{\partial b_0}{\partial y_0} \right) [\lambda] = 0 \qquad (3.12)$$

which is an equation for the prevalence of discontinuities $[\lambda] = \lambda^+ - \lambda^-$ on the characteristics. Equation (3. 12) was obtained by using formula (A. 15) for Eq. (2. 6).

Let us obtain another Weierstrass — Erdmann condition. For this we consider the $(n-1)$-dimensional surface C which determines the two characteristics $y_0^{(1)} = \text{const}$ and $y_0^{(2)} = \text{const}$ (see A. 18) which form four elementary domains abutting C. Equating the coefficient of the variation $\Delta z|_C$ in (3. 5) to zero, we obtain

$$\sum_{k=1}^{n}\sum_{l=1}^{4}(-1)^l \lambda_l \left[\frac{b_k^{(1)}\cos(N_1, y_k)}{R_0^{(1)}R_1^{(1)}\ldots R_n^{(1)}} + \frac{b_k^{(2)}\cos(N_2, y_k)}{R_0^{(2)}R_1^{(2)}\ldots R_n^{(2)}}\right] = 0 \qquad (3.13)$$

where by λ_l $(l = 1, 2, 3, 4)$ we denote the value of λ on surface C but belonging to each of the four elementary domains. From formulas (A. 18) and (A. 19) follows

$$b_k^{(1)} = -b_k^{(2)},\ \cos(N_1, y_k) = -\cos(N_2, y_k),\ R_i^{(1)} = R_i^{(2)} \qquad (3.14)$$

Then condition (3. 13) becomes

$$\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 = 0 \text{ on } C$$

A similar condition was obtained in [2] for the two-dimensional case. It shows that the magnitude of the discontinuity in the multiplier λ does not change on passing through the other characteristic.

4. **Boundary conditions.** To obtain the conditions on the boundary S of domain W we should use the quantities from expression (2. 2) which depend on the coordinates of the boundaries. First of all we consider an elementary domain w having a boundary in common with S

$$\delta I_s = \int_s \left(\nu\Delta z_N + \frac{\partial h}{\partial z}\Delta z + \sum_{i=1}^{q}\frac{\partial h}{\partial v_i}\Delta v_i\right)ds + \int_s \left[b_0\lambda\Delta\frac{\partial z}{\partial y_0} + F(\lambda)\Delta z\right]\frac{dy}{R_0R_1\ldots R_n} + \int_C \sum_{k=1}^{n} b_k\cos(N', y_k)\frac{\lambda\Delta z}{R_0R_1\ldots R_n}dC \qquad (4.1)$$

We assume here that s is given by the equation $y_0(x_0, x) = D$; therefore, $\partial z/\partial N = R_0\partial z/\partial y_0$ and $ds = K_1K_2\ldots K_n dy$. Terms for the variations δy_0 and δN are absent in (4. 1) since the surfaces are assumed fixed and $\delta y_0 = \delta N = 0$. Equating the coefficients of variations $\Delta\partial z/\partial y_0$, Δz and Δv_i to zero, on the boundary s of each elementary domain we obtain the conditions

$$R_0^2\nu + b_0\lambda = 0,\quad F(\lambda) + R_0\frac{\partial h}{\partial z} = 0,\quad \frac{\partial h}{\partial v_i} = 0,\quad i = 1, \ldots, q \qquad (4.2)$$

which are the boundary conditions to Eq. (2. 6) and serve as well for the determination of multiplier ν and boundary controls v_i.

Further, we obtain conditions on the $(n-1)$-dimensional surfaces C lying on boundary S; such a surface defines two characteristics which form three elementary domains lying in W and abutting C. We number these elementary domains by indices 1, 2 and 3 so that the elementary domain with index 2 is located between the two characteristics. Equations the coefficient of variation $\Delta z|_C$ to zero, we obtain the condition

$$\sum_{k=1}^{n}\left[\frac{\lambda_1 - \lambda_2}{R_0^{(1)}R_1^{(1)}\ldots R_n^{(1)}}b_k^{(1)}\cos(N_1', y_k) - \right. \qquad (4.3)$$

$$\left.\frac{\lambda_2-\lambda_3}{R_0^{(2)}R_1^{(2)}\dots R_n^{(2)}}\,b_k^{(2)}\cos(N_2',y_k)\right]=0$$

Using formulas (3.14), from equality (4.3) we finally obtain the condition

$$\lambda_1-2\lambda_2+\lambda_3=0 \text{ on } C \tag{4.4}$$

where λ_2 is the value of the Lagrange multiplier λ in the elementary domain with index 2.

Let us now find the conditions on the hypersurfaces Ω for $x_0 = T$. For this we write down the terms from the first variation, depending on the coordinates of this hypersurface

$$\delta I_T=\int_\omega\left(\frac{\partial\varphi_0}{\partial z_{x_0}}\Delta\frac{\partial z}{\partial x_0}+\frac{\partial\varphi_0}{\partial z}\Delta z\right)dx+\int_\Gamma\sum_{i=0}^{r}\frac{\partial\chi}{\partial z}\Big|_{\Gamma_i}\Delta z\,|_{\Gamma_i}\,d\Gamma+ \tag{4.5}$$

$$\int_\omega\left[b_0\lambda\Delta\frac{\partial z}{\partial y_0}+F(\lambda)\Delta z\right]\frac{dy}{R_0R_1\dots R_n}+\int_C\sum_{k=1}^{n}\times$$

$$b_k\cos(N',y_k)\frac{\lambda\Delta z dC}{R_0R_1\dots R_n}$$

The boundary being examined can be given by the equation $y_0 = x_0 = T$; therefore, we can also set $y_k = x_k$, $k = 1, \dots, n$. Taking what has been said into account, we obtain

$$R_0=R_k=b_0=1;\quad b_k=c_0=c_0'=\frac{\partial b_0}{\partial y_0}=\frac{\partial^2 y_0}{\partial x_0^2}=\frac{1}{\rho_{0k}}=0, \tag{4.6}$$
$$k=1,\dots,n;\quad d_0=a_0$$

Equating the coefficients of the variations $\Delta\partial z/\partial x_0$ and Δz to zero, we have

$$\lambda+\frac{\partial\varphi_0}{\partial z_{x_0}}=0,\quad -\frac{\partial\lambda}{\partial x_0}+a_0\lambda+\frac{\partial\varphi_0}{\partial z}=0 \tag{4.7}$$

Let us obtain more conditions on the $(n-1)$-dimensional surfaces C lying on Ω for $x_0 = T$. In this case surfaces C can be of two types: coinciding with the $(n-1)$-dimensional surfaces $\Gamma_0, \Gamma_1, \dots, \Gamma_r$ and not coinciding with even one of the Γ_i, $i = 0, 1, \dots, r$. First of all we consider the surfaces Γ_i, $i=1, \dots, r$. A surface Γ_i defines two families of characteristics $y_0^{(1)} = \text{const}$ and $y_0^{(2)} = \text{const}$ which divide domain W adjoining $x_0 = T$ into three elementary domains. We denote these elementary domains by indices 1, 2 and 3, where the index 2 denotes the elementary domain formed by the two characteristics. Then, equating the expression for the variation $\Delta z|_{\Gamma_i}$ to zero, we obtain a condition analogous to (4.3) with the additional term $\partial\chi/\partial z|_{\Gamma_i}$ on the left-hand side. Using formulas (3.14), we can transform this condition to the form

$$\left[\frac{\lambda_1-2\lambda_2+\lambda_3}{R_0^{(1)}R_1^{(1)}\dots R_n^{(1)}}\sum_{k=1}^{n}b_k^{(1)}\cos(N_1',x_k)+\frac{\partial\chi}{\partial z}\Big|_{\Gamma_i}\right]_{\Gamma_i}=0,\quad i=1,\dots,z \tag{4.8}$$

Analogously we can obtain conditions for the $(n-1)$-dimensional surfaces C lying on Ω but not coinciding with the Γ_i, $i = 0, 1, \dots, r$. The sole difference is the the absence of the term $\partial\chi/\partial z|_{\Gamma_i}$ in the expression similar to (4.8)

$$\lambda_1-2\lambda_2+\lambda_3=0 \text{ on } C \tag{4.9}$$

We now consider the $(n-1)$-dimensional surface Γ_0, being a boundary of domain

Ω or the intersection of the cylindrical hypersurface S and $x_0 = T$. Surface Γ_0 defines one characteristic lying in domain W and separating domain W adjacent to Γ_0 into two elementary domains. We denote the quantities belonging to these elementary domains by indices 1 and 2, respectively. Equating the expression for the variation $\Delta z\,|_{\Gamma_0}$ to zero, we obtain the condition

$$\left[(\lambda_1 - \lambda_2)\sum_{k=1}^{n} b_k \cos(N', x_k) + R_0 R_1 \ldots R_n \frac{\partial \chi}{\partial z}\Big|_{\Gamma_0}\right]_{\Gamma_0} = 0 \tag{4.10}$$

Conditions (4.7) and (4.10) are terminal conditions for Eq. (2.6). In the general case these terminal conditions are discontinuous; therefore, the solution of Eq. (2.6) has discontinuities on the characteristics generated by the ($n-1$)-dimensional surfaces lying in domain Ω for $x_0 = T$.

We obtain further conditions on the hypersurface Ω for $x_0 = t_0$. From the first variation we write out the terms depending on the coordinates of this hypersurface

$$\delta I_{t_0} = \int_{\omega} [\nu_1 \Delta z(t_0, x) + \nu_2 \Delta z_{x_0}(t_0, x)]\, dx + \int_{\omega} \Big[b_0 \lambda \Delta \frac{\partial z}{\partial y_0} +$$

$$F(\lambda)\, \Delta z\Big] \frac{dy}{R_0 R_1 \ldots R_n} + \int_{C} \sum_{k=1}^{n} b_k \cos(N', y_k) \frac{\lambda \Delta z dC}{R_0 R_1 \ldots R_n}$$

Relations (4.6) are valid on the hypersurface $x_0 = t_0$; therefore, equating the quantities with Δz_{x_0} and Δz to zero, we obtain the conditions

$$\nu_2 + \lambda = 0, \quad \nu_1 - \frac{\partial \lambda}{\partial y_0} + a_0 \lambda = 0$$

which serve to determine the Lagrange multipliers $\nu_1(x)$ and $\nu_2(x)$. The inequalities

$$\lambda_1 - 2\lambda_2 + \lambda_3 \neq 0 \quad \text{on } C$$

may be valid on the ($n-1$)-dimensional surfaces; they show that the discontinuities of the Lagrange multiplier, started when $x = T$, can reach the boundaries $x_0 = t_0$.

5. The Weierstrass necessary condition. We pass to the proof of the Weierstrass necessary conditions. For the case of two independent variables the proof was obtained in [3, 4]. For $n+1$ independent variable the proof proceeds analogously, and so is not presented here in detail. The Weierstrass necessary conditions include a condition at the interior points of domain W and a condition at points on boundary S.

Let us prove the Weierstrass condition at the interior points. Suppose that we have the solution u, v, z minimizing functional (1.7). By $B(x^\circ, r_0)$ we denote a closed n-dimensional sphere and by $\sigma(x^\circ, r_0)$ we denote the surface of this sphere of radius r_0 with its center at point x°. Let us consider the cylinder $B(x^\circ, r_0) \cup (0 \leqslant x_0 - x^\circ_0 \leqslant e)$ lying in an arbitrary elementary domain w In this cylinder we construct a new control U satisfying relations (2.2). Then the solution of Eq. (1.1) differs from the optimal in the domain bounded by sphere $B(x^\circ, r_0)$, $x_0 = x_0{}^\circ$ and the characteristic conoid [6] passing through the sphere's surface $\sigma(x^\circ, r_0)$ for $x_0 \geqslant x^\circ$. Further, by implementing transformations analogous to those in [3], we can obtain the Weierstrass necessary condition

$$H(x_0, x, z, U, \lambda, \mu) - H(x_0, x, z, u, \lambda, \mu) \geqslant 0 \tag{5.1}$$

which must be satisfied at each interior point of the elementary domains.

To prove the Weierstrass conditions for boundary controls we consider the solution u, v, z, minimizing functional J. We consider the intersection of the cylinder $B(x^\circ, r_0) \bigcup (0 \leqslant x_0 - x_0^\circ \leqslant e)$ and the lateral hypersurface S belonging to the boundary of elementary domain w. In this intersection we choose a control V satisfying relations (1.5). Then the solution of Eq. (1.1) differs from the optimal in a domain bounded by the characteristic hypersurfaces passing through the ($n-1$)-dimensional surface $(B(x^\circ, r_0), x_0 = x_0^\circ) \bigcap S$ for $x_0 > x_0^\circ$. Further, by implementing constructions analogous to those in [4], we can obtain the second Weierstrass necessary condition

$$h(S, z|_S, V, \nu, \varkappa) - h(S, z|_S, v, \nu, \varkappa) \geqslant 0 \tag{5.2}$$

Here and in condition (5.1) u and v are optimal controls, while U and V are arbitrary controls satisfying relations (1.2) and (1.5); therefore, they imply the Weierstrass necessary conditions for a strong minimum.

Appendix. Let us consider the differential equation of hyperbolic type [7]

$$L(z) = \frac{\partial^2 z}{\partial x_0^2} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 z}{\partial x_i \partial x_j} + \sum_{i=0}^{n} a_i \frac{\partial z}{\partial x_i} = f(x_0, x, u, z) \tag{A.1}$$

In the space R^{n+1} of variables $x_0, x_1, \ldots, x_n$ we take the mutually orthogonal family of surfaces

$$y_0 = y_0(x_0, x),\ y_1 = y_1(x_0, x),\ \ldots,\ y_n = y_n(x_0, x) \tag{A.2}$$

We introduce the curvilinear coordinates $y_0, y_1, \ldots, y_n$. Solving Eqs. (A.2) relative to $x_0, x_1, \ldots, x_n$, we obtain

$$x_0 = x_0(y_0, y),\ x_1 = x_1(y_0, y),\ \ldots,\ x_n = x_n(y_0, y) \tag{A.3}$$

By $\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_n$ we denote the orthonormalized frame connected with the coordinates $x_0, x_1, \ldots, x_n$ and by $\mathbf{e}_0', \mathbf{e}_1', \ldots, \mathbf{e}_n'$ we denote the unit vectors of the moving frame connected with the coordinates $y_0, y_1, \ldots, y_n$ [8]. Then we obtain

$$\mathbf{e}_i' = \frac{1}{K_i} \sum_{j=0}^{n} \frac{\partial x_j}{\partial y_i} \mathbf{e}_j, \qquad K_i = \left[\sum_{j=0}^{n} \left(\frac{\partial x_j}{\partial y_i}\right)^2\right]^{1/2}, \qquad i = 0, 1, \ldots, n \tag{A.4}$$

The matrix α, set by the elements $\alpha_{ij} = K_i^{-1} \partial x_j / \partial y_i$, where i is the row number and j is the column number, corresponds to a unitary operator, therefore, $\alpha^{-1} = \alpha^T$, where α^T denotes the transpose of matrix α. Using this important property of matrix α, we can obtain the following formulas

$$\frac{\partial x_j}{\partial y_i} = \frac{1}{R_i^2} \frac{\partial y_i}{\partial x_j}, \qquad \frac{\partial y_i}{\partial x_j} = \frac{1}{K_i^2} \frac{\partial x_j}{\partial y_i}, \qquad R_i = \left[\sum_{j=0}^{n} \left(\frac{\partial y_i}{\partial x_j}\right)^2\right]^{1/2} \tag{A.5}$$

$$R_i = \frac{1}{K_i}, \qquad \frac{1}{R_i R_j} \sum_{p=0}^{n} \frac{\partial y_i}{\partial x_p} \frac{\partial y_j}{\partial x_p} = \sum_{p=0}^{n} \frac{1}{R_p^2} \frac{\partial y_p}{\partial x_i} \frac{\partial y_p}{\partial x_j} = \delta_{ij}, \tag{A.6}$$

$$i, j = 0, 1, \ldots, n$$

where δ_{ij} is the Kronecker symbol.

The differential arc length ds in the new coordinates is

$$ds^2 = \sum_{i=0}^{n} (K_i dy_i)^2 = \sum_{i=0}^{n} \left(\frac{dy_i}{R_i}\right)^2 \tag{A.7}$$

In what follows we shall also need the derivatives of the unit vectors $\mathbf{e}_p'$ with respect to the variables y_r. From the expressions for the derivatives

$$\frac{\partial \mathbf{e}_p'}{\partial y_r} \mathbf{e}_q' = 0, \quad r \neq p, \quad r \neq q, \quad p \neq q; \frac{\partial \mathbf{e}_p'}{\partial y_q} \mathbf{e}_q' + \frac{\partial \mathbf{e}_q'}{\partial y_q} \mathbf{e}_p' = 0, \quad p \neq q \tag{A.8}$$

we can obtain the formulas

$$\sum_{i,j=0}^{n} \frac{\partial^2 y_p}{\partial x_i \partial x_j} \frac{\partial y_q}{\partial x_i} \frac{\partial y_z}{\partial x_j} = 0, \quad r \neq p, \quad r \neq q, \quad p \neq q \tag{A.9}$$

$$\frac{1}{\rho_{pq}} = \frac{1}{K_q} \frac{\partial e_q'}{\partial y_q} e_p' = -\frac{1}{K_p K_q^2} \sum_{j=0}^{n} \frac{\partial^2 x_j}{\partial y_p \partial y_q} \frac{\partial x_j}{\partial y_q}, \quad p \neq q$$

$$\frac{1}{\rho_{pq}} = -\frac{1}{R_p R_q^2} \sum_{i,j=0}^{n} \frac{\partial^2 y_p}{\partial x_i \partial x_j} \frac{\partial y_q}{\partial x_i} \frac{\partial y_q}{\partial x_j}, \quad p \neq q$$

where $1/\rho_{pq}$ denotes the curvature of the line defined by the unit vector e_q', lying on the hypersurface $y_0(x_0, x_1, \ldots, x_n) = \text{const}$ (Dupin's theorem [9]). Analogously, by $1/\rho_{pp}$ we denote the value in the last two formulas, resulting when $q = p$.

Now let us pass to the new variables $y_0, y_1, \ldots, y_n$ in Eq. (A.1), and consider it on the hypersurface $y_0(x_0, x_1, \ldots, x_n) = \text{const}$. Then we obtain the equation

$$L(z) = b_0 \frac{\partial^2 z}{\partial y_0^2} + 2\sum_{k=1}^{n} b_k \frac{\partial^2 z}{\partial y_0 \partial y_k} + \sum_{k=0}^{n} \frac{\partial^2 y_k}{\partial x_0^2} \frac{\partial z}{\partial y_k} - \tag{A.10}$$

$$\sum_{k,l=1}^{n} b_{kl} \frac{\partial^2 z}{\partial y_k \partial y_l} - \sum_{k=0}^{n} c_k \frac{\partial z}{\partial y_k} + \sum_{k=0}^{n} d_k \frac{\partial z}{\partial y_k} = f$$

in which we have denoted

$$b_p = \frac{\partial y_0}{\partial x_0} \frac{\partial y_p}{\partial x_0} - \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_0}{\partial x_i} \frac{\partial y_p}{\partial x_j}, \quad b_{kl} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j}$$

$$c_p = \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 y_p}{\partial x_i \partial x_j}, \quad d_p = \sum_{i=0}^{n} a_i \frac{\partial y_p}{\partial x_i}; \quad k, l = 1, \ldots, n; \quad p = 0, 1, \ldots, n$$

In the new coordinates in Eq. (A.10) the derivatives $\partial z / \partial y_0$, $\partial^2 z / \partial y_0^2$ and $\partial^2 z / \partial y_0 \partial y_i$ are computed along the normal, while the remaining derivatives, along the tangent to the surface $y_0(x_0, x) = \text{const}$. We assume that function z is continuous together will all its derivatives when passing through the hypersurface $y_0(x_0, x) = \text{const}$. Then the derivatives $\partial^2 z / \partial y_0 \partial y_i$ and $\partial^2 z / \partial y_i \partial y_j$ remain continuous. For $\partial^2 z / \partial y_0^2$ we have

$$b_0 \left[\frac{\partial^2 z}{\partial y_0^2}\right] = [f], \quad \left[\frac{\partial^2 z}{\partial y^2}\right] = \frac{\partial^2 z^+}{\partial y_0^2} - \frac{\partial^2 z^-}{\partial y_0^2}, \quad [f] = f^+ - f^- \tag{A.11}$$

Consequently, $[\partial^2 z / \partial y_0^2] \neq 0$ if $[f] \neq 0$ and the hypersurface $y_0(x_0, x) = \text{const}$ is not a solution of the differential equation

$$b_0 = \left(\frac{\partial y_0}{\partial x_0}\right)^2 - \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_0}{\partial x_i} \frac{\partial y_0}{\partial x_j} = 0 \tag{A. 12}$$

i. e. is not a characteristic of Eq. (A. 1). When the right-hand side is continuous, the derivative $\partial^2 z / \partial y_0^2$ can have finite or infinite discontinuities only on a characteristic.

We differentiate Eq. (A. 10) with respect to y_0 under the condition that $y_0(x_0, x) =$ const is a characteristic. Then we obtain the equation

$$2\sum_{k=1}^{n} b_k \frac{\partial}{\partial y_k}\left[\frac{\partial^2 z}{\partial y_0^2}\right] + \left(\frac{\partial^2 y_0}{\partial x_0^2} - c_0 + d_0 + \frac{\partial b_0}{\partial y_0}\right)\left[\frac{\partial^2 z}{\partial y_0^2}\right] = \left[\frac{\partial f}{\partial y_0}\right] \tag{A. 13}$$

which is called the equation for the prevalence of second-order discontinuities. If we assume the continuity of function z and the discontinuity of the derivative $\partial z / \partial y_0$ along $y_0(x_0, x) =$ const, it is clear that $y(x_0, x) =$ const is a characteristic, while the quantity $[\partial z / \partial y_0] = \partial z^+ / \partial y_0 - \partial z^- / \partial y_0$ satisfies the differential equation

$$2\sum_{k=1}^{n} b_k \frac{\partial}{\partial y_k}\left[\frac{\partial z}{\partial y_0}\right] + \left(\frac{\partial^2 y_0}{\partial x_0^2} - c_0 + d_0\right)\left[\frac{\partial z}{\partial y_0}\right] = [f] \tag{A. 14}$$

If, however, function z is discontinuous, then for the magnitude of the discontinuity $[z] = z^+ - z^-$ we obtain the equation

$$2\sum_{k=1}^{n} b_k \frac{\partial}{\partial y_k}[z] + \left(\frac{\partial^2 y_0}{\partial x_0^2} - c_0 + d_0 - \frac{\partial b_0}{\partial y_0}\right)[z] = 0 \tag{A. 15}$$

Equations (A. 13) and (A. 14) show that the discontinuities $[\partial^2 z / \partial y_0^2]$ and $[\partial z / \partial y_0]$ can arise both at the expense of boundary conditions as well at the expense of discontinuities $[\partial f / \partial y_0]$ and $[f]$. The discontinuities $[z]$ can arise only at the expense of discontinuities $[z]$ in the boundary or initial conditions.

Let us analyze Eq. (A. 12). Note first of all that its dimensionality can be lowered by a change of variables

$$y_0 = x_0 \pm Y(x_1, \ldots, x_n) = x_0 \pm Y(x) \tag{A. 16}$$

In this case Eq. (A. 12) takes the form

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial Y}{\partial x_i} \frac{\partial Y}{\partial x_j} = 1 \tag{A. 17}$$

From (A. 16) it follows that if we solve Eq. (A. 12) with initial conditions on an $(n-1)$-dimensional surface C, $y_0|_C = D$, we can obtain two solutions

$$y_0^{(1)} = x_0 + Y(x) = D, \quad y_0^{(2)} = x_0 - Y(x) = D \tag{A. 18}$$

Therefore, we can assert that each $(n-1)$-dimensional surface C defines two characteristics (A. 18) which separate the domain of variables x_0 and x into four parts.

We should note another important property of hypersurfaces (A. 18). Consider the normal N_1 to surface C, lying in the tangent hyperplane to the hypersurface $y_0^{(1)}$, and the normal N_2 lying in the tangent hyperplane to the hypersurface $y_0^{(2)}$. Then from (A. 18) follows

$$\cos(N_1, x_i) = \cos(N_2, -x_i) = \cos[(N_2, x_i) + \pi] = -\cos(N_2, x_i)$$

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